INTERSECTION COHOMOLOGY OF S^1 SYMPLECTIC QUOTIENTS AND SMALL RESOLUTIONS

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Abstract. We provided two explicit formulas for the intersection cohomology (as a graded vector space with pairing) of the symplectic quotient by a circle in terms of the S^1 equivariant cohomology of the original symplectic manifold and the fixed point data. The key idea is the construction of a small resolution of the symplectic quotient.

1. Introduction

Let a compact Lie group act effectively on a compact connected symplectic manifold M with a moment map $\Phi : M \to \mathfrak{g}^*$. In the case that 0 is a regular value of the moment map, the symplectic quotient $M_{\text{red}} := \Phi^{-1}(0)/G$ is an orbifold, and its rational cohomology ring is fairly well understood [Ki2, Wi, Wu, Ka, GK, JK, TW].

However, many interesting spaces arise as reduced spaces at singular values of the moment map. Some examples include: the moduli space of flat connections, some polygon spaces, many physical systems, and projective toric varieties whose polytopes are not simple. Since the symplectic quotient at a singular value is a stratified space [SL], a natural invariant to compute is the intersection cohomology (with middle perversity). Less is known in this case. Kirwan has provided formulas to compute the Betti numbers in the algebraic case [Ki2, Ki3]; Woolf extended this work to the symplectic case. Moreover, Jeffrey and Kirwan computed the pairing in the intersection cohomology of particular symplectic quotients [Ki4].

The main result of this paper is two explicit formulas for the intersection cohomology (as a graded vector space with pairing) of the symplectic quotient by a circle in terms of the S^1 equivariant cohomology of the original symplectic manifold and the fixed point data. More precisely, these formulas depend on the image of the restriction map in equivariant cohomology $H^*_{S^1}(M;\mathbb{R}) \to H^*_{S^1}(M^{S^1};\mathbb{R})$ from the original manifold to the fixed point set.

Theorem 1. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of $\Phi(M)$. Let $M_{red} := \Phi^{-1}(0)/S^1$ denote the reduced space.*

Then there exists a surjective map κ from the equivariant cohomology ring $H^*_{S^1}(M;\mathbb{R})$ to the intersection cohomology $IH^*(M_{red};\mathbb{R})$. Moreover, given any equivariant cohomology class α and β in $H^*_{S^1}(M)$, the pairing of $\kappa(\alpha)$ and $\kappa(\beta)$ in $IH^*(M_{red})$ is given by the formula

$$\langle \kappa(\alpha), \kappa(\beta) \rangle = \operatorname{Res}_0 \sum_{F \in \mathcal{F}^+} \int_F \frac{i_F^*(\alpha\beta)}{e_F}.$$

Here, e_F denotes the equivariant Euler class of the normal bundle of F, and $\mathcal{F}^+$ denotes the set of components F of the fixed point set S^1 such that either

1. *$\Phi(F) > 0$ or*

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2. $\Phi(F) = 0$ *and* $\operatorname{index} F \leq \frac{1}{2}(\dim M - \dim F)$,

where the index of F is the dimension of the negative eigenspace of the Hessian of the moment map Φ at a point of F.

The meaning of the right hand side is as follows. The map i_F^* is simply the restriction to F. The equivariant cohomology ring $H^*_{S^1}(F)$ is naturally isomorphic to the polynomial ring in one variable $H^*(F)[t]$. The equivariant Euler class e_F is invertible in the localized ring $H^*(F)(t)$; thus, $\frac{i_F^*(\alpha)}{e_F}$ is an element of this ring. The integral $\int_F : H^*(F)(t) \to \mathbb{R}(t)$ acts by integrating each coefficient in the series. Finally, Res_0 denotes the operator which returns the coefficient of t^{-1}.

Remark 1.1. Our convention is that the pairing in intersection cohomology between two classes $\alpha \in IH^p(M_{\rm red})$ and $\beta \in IH^q(M_{\rm red})$ is zero if $p + q \neq \dim(M_{\rm red})$.

Note that since κ is surjective, this theorem determines the pairing for all pairs of elements in $IH^*(M_{\rm red})$. Additionally, by Poincare duality in intersection cohomology, it determines the kernel of κ.

We now provide an alternative version of our main result:

Theorem 1$'$. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of $\Phi(M)$. Let $M_{red} := \Phi^{-1}(0)/S^1$ denote the reduced space.*

Then there exists a ring structure on the intersection cohomology $IH^*(M_{red}; \mathbb{R})$ so that

- *The ring structure on $IH^*(M_{red})$ is compatible with the pairing, in the sense that their exists an isomorphism $\int$ from the top dimensional intersection cohomology to $\mathbb{R}$ so that $\int \alpha \cdot \beta = \langle \alpha, \beta \rangle$.*
- *As a graded ring, $IH^*(M; \mathbb{R})$ is isomorphic to $H^*_{S^1}(M; \mathbb{R})/K$, where*

$$K := \{\alpha \in H^*_{S^1}(M) \mid \alpha|_F = 0 \ \ \forall \ \ F \in \mathcal{F}^+\} \oplus \{\alpha \in H^*_{S^1}(M) \mid \alpha|_F = 0 \ \ \forall \ \ F \in \mathcal{F}^-\}.$$

Here, $\mathcal{F}^+$ denotes the set of components F of the fixed point set M^{S^1} such that either

1. $\Phi(F) > 0$ *or*
2. $\Phi(F) = 0$ *and* $\operatorname{index} F \leq \frac{1}{2}(\dim M - \dim F)$,

where the index of F is the dimension of the negative eigenspace of the Hessian of the moment map Φ at a point of F. Additionally, $\mathcal{F}^-$ denotes the set of all other components of the fixed point set.

In principle, these two formulas for the intersection cohomology give almost exactly the same information. We include both, because, in practice, one or the other might be better suited to tackle a particular problem.

We prove these two theorems simultaneously. First, we construct an orbifold $\widetilde{M}_{\rm red}$, which we call the **perturbed quotient**, The perturbed quotient is a small resolution of the symplectic quotient; thus. as a graded vector space with pairing, $IH^*(\Phi^{-1}(0)/S^1)$ is isomorphic to $H^*(\widetilde{M_{\rm red}})$. Moreover, even though the perturbed quotient is not symplectic, it is constructed in such a way that the standard techniques for computing the cohomology ring of a symplectic quotient can be applied to it, yielding the above formulae.

The construction of the perturbed quotient is fairly straightforward. The singularities of the reduced space $M_{\rm red} := \Phi^{-1}(0)/S^1$ correspond to components Y of the fixed point set M^{S^1} lying on the zero level set $\Phi^{-1}(0)$. In the setting projective varieties, it is known that the neighborhoods of these singularities have small resolutions [H].[1] Although these resolutions are only local, it is possible to piece them together into a global resolution.

[1] It is claimed in [H] that the resolutions are global, which need not be the case; see [H2],

We construct the perturbed quotient as the quotient of a fiber of a perturbation $\tilde{\Phi} : M \to \mathbb{R}$ of the original moment map. Since this perturbed moment map $\tilde{\Phi}$ is Bott-Morse, and since its critical points are exactly the fixed points of the action of S^1 on M, the standard techniques used to compute the cohomology of symplectic quotients can also be applied to compute the cohomology of the perturbed quotient.

Finally, we construct a pairing preserving isomorphism between the intersection cohohomology of the symplectic quotient and the cohomology of the perturbed quotient. In the algebraic case, this follows immediately from the fact that the perturbed quotient is a small resolution of the symplectic quotient (see §6.2 in [GM]). However, while it "appears to be clear that this theorem will also be valid in our case", we have decided to provide a direct proof.

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2. Simple stratified spaces and intersection cohomology

In this section, we introduce the two main concepts that we will need in this paper: simple stratified spaces and intersection cohomology. The notion of a simple stratified space is not standard; it is, however, convenient for our purposes. The definition of intersection cohomology we use is essentially identical to the definition of the complex of intersection differential forms due to Goresky and MacPherson (see [B]), except that we allow the strata to be orbifolds, and that we only consider simple stratified spaces.

Recall that an open **cone** on a topological space L is

$$\overset{\circ}{c}(L) := L \times [0,1)/\sim,$$

where $(x,0) \sim (x',0)$ for all $x, x' \in L$. Equivalently $\overset{\circ}{c}(L) = L \times [0,\infty)/\sim$.

Definition 2.1. A **simple stratified space** is a Hausdorff topological space X with the following properties:

- The space X is a disjoint (set-theoretic) union of orbifolds, called **strata**.
- There exists an open dense oriented stratum X^r, called the **top stratum**.
- The complement of X^r in X is a disjoint union of connected orbifolds, $X \smallsetminus X^r = \coprod Y_i$, called the **singular strata**.
- For each singular stratum Y there is a neighborhood $\tilde{T}$ of Y in X and a map $\pi : \tilde{T} \to Y$ which is a C^0 fiber bundle with a typical fiber $\overset{\circ}{c}(L)$ for some orbifold L, which depends on Y. (Thus Y embeds into $\tilde{T}$ as the vertex section.)
- Their exists a diffeomorphism from the complement $\tilde{T} \smallsetminus Y$ to $Q \times (0,1)$, where $Q \to Y$ is a C^∞ fiber bundle with typical fiber L, such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{T} \smallsetminus Y & \longrightarrow & Q_i \times (0,1) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ Y & = & Y. \end{array}$$

In particular $\pi : \tilde{T} \smallsetminus Y \to Y$ is a smooth fiber bundle with a typical fiber $L \times (0,1)$.

Thus a simple stratified space X is a decomposition $X = X^r \sqcup \coprod Y_i$ and a collection of maps $\{\pi_i : \tilde{T}_i \to Y_i\}$.

Remark 2.2. Note that the composite $\tilde{T} \smallsetminus Y \to Q \times (0,1) \to (0,1)$, where $Q \times (0,1) \to (0,1)$ is the obvious projection, extends to a continuous map $r : \tilde{T} \to [0,1)$. In the definition of intersection cohomology of X it will be convenient for us to consider smaller tubular neighborhoods T defined by

$$T = r^{-1}([0,1/2)).$$

Definition 2.3 (Cartan filtration of forms relative to a submersion)**.** Let $\pi : E \to B$ be a smooth submersion of orbifolds. The Cartan filtration $\mathbb{F}_k\Omega^*(E)$ of the complex of forms $\Omega^*(E)$ on E is given by

$$\begin{aligned}\mathbb{F}_k\Omega^*(E) = \{\omega \in \Omega^*(E) \mid\ & \text{for all } \ e \in E \text{ and for all vectors } \xi_0, \cdots, \xi_k \in \ker d\pi_e \\ & i(\xi_0) \circ \cdots \circ i(\xi_k)(\omega(e)) = 0 \\ & \text{and } i(\xi_0) \circ \cdots \circ i(\xi_k)(d\omega(e)) = 0\}.\end{aligned}$$

By convention, $i(\xi_0) \circ \cdots \circ i(\xi_k)(\sigma) = 0$ if $\deg \sigma \leq k$. Note that $\mathbb{F}_0\Omega^*(E)$ consists of basic forms.

Let $X = X^r \sqcup \coprod Y_i$ be a simple stratified space. A **perversity** $\bar{p} : \{Y_i\} \to \mathbb{N}$ is a function that assigns a nonnegative integer to each singular stratum Y_i. The **middle perversity** $\overline{m}$ is defined by

$$\bar{m}(Y_i) = \lfloor \frac{1}{2}(\dim X^r - \dim Y_i) \rfloor - 1.$$

Definition 2.4. Let $(X = X^r \sqcup \coprod Y_i, \{\pi_i : \tilde{T}_i \to Y_i\})$ be a simple stratified space and let $\overline{p} : \{Y_i\} \to \mathbb{N}$ be a perversity. The **complex of intersection differential forms** $I\Omega^*_{\overline{p}}(X)$ is a sub-complex of the complex of differential forms on the top stratum:

$$I\Omega^*_{\overline{p}}(X) := \{\omega \in \Omega^*(X^r) \mid \omega|_{T_i \cap X^r} \in \mathbb{F}_{\overline{p}(Y_i)}\Omega^*(T_i \cap X^r)\}$$

where the filtration $\mathbb{F}_{\overline{p}(Y_i)}\Omega^*(T_i \cap X^r)$ is defined relative to the submersion $\pi_i : T_i \cap X^r = T_i \smallsetminus Y_i \to Y_i$. The coboundary map is the exterior differentiation d.

The **intersection cohomology** $IH^*_{\bar{p}}(X)$ of the simple stratified space X with perversity $\bar{p}$ is the cohomology of the complex $(I\Omega^*_{\bar{p}}(X), d)$.

Remark 2.5. If the strata of a simple stratified space X are manifolds, then X is a pseudomanifold. In this case our definition of intersection forms is exactly the Goresky-MacPherson definition of the complex of differential forms and our definition of intersection cohomology agrees with the standard definition (see § 1.2 in [B]).

We now define the pairing on the middle perversity intersection cohomology of a compact simple stratified space X with an oriented top stratum.

Note that if $q > \dim(Y_i) + \overline{p}(Y_i)$, then every $\alpha \in I\Omega^q_{\overline{p}}(X)$ vanishes on T_i. In particular, if $\dim X^r > \dim Y_i + \overline{p}(Y_i)$ for all singular strata Y_i then every $\alpha \in I\Omega^{\dim X^r}_{\overline{p}}(X)$ is supported on the compact set $X \smallsetminus \bigcup T_i$. Therefore, if the top stratum X^r is oriented, there is a well-defined integration map $\int : I\Omega^{\dim X^r}_{\overline{p}}(X) \to \mathbb{R}$, $\alpha \mapsto \int_{X^r} \alpha$. Similarly, if $\dim X^r - 1 > \dim Y_i + \overline{p}(Y_i)$ for all i, then any $\beta \in I\Omega^{\dim X^r - 1}_{\overline{p}}(X)$ is also supported in $X \smallsetminus \bigcup T_i$. Thus, integration descends to a well-defined map on cohomology $\int : IH^{\dim X^r}_{\overline{p}}(X) \to \mathbb{R}$; we extend this by zero to a map $\int : IH^*_{\overline{p}}(X) \to \mathbb{R}$.

Given α and β in $I\Omega^*_{\overline{m}}(X)$, notice that $\alpha\wedge\beta \in I\Omega_{2\overline{m}}(X)$, where $2\overline{m}$ is twice the middle perversity. This follows from the property of the Cartan filtration: for any $\alpha \in \mathbb{F}_k\Omega^*(E)$ and $\beta \in \mathbb{F}_l\Omega^*(E)$, $\alpha \wedge \beta \in \mathbb{F}_{k+l}\Omega^*(E)$. Moreover, $2\overline{m}(Y_i) \leq \dim X^r - \dim Y_i - 2$ for all singular strata Y_i. Thus, there is a well-defined bilinear pairing $IH^p_{\overline{m}}(X) \times IH^q_{\overline{m}}(X) \to \mathbb{R}$ which sends $[\alpha] \in IH^p_{\overline{m}}(X)$ and $[\beta] \in IH^q_{\overline{m}}(X)$ to the integral $\int_{X^r} \alpha \wedge \beta$.

3. The structure of the symplectic quotient

In this section, we recall a normal form for the neighborhoods of fixed points on symplectic manifolds with Hamiltonian circle actions. Using this, we give a normal form for the neighborhoods of the singularities in a symplectic quotient. In particular, we show that the quotient is a simple stratified space.

This last statement is a special case of a theorem of Sjaamar and Lerman [SL], who show that every symplectic quotient by a compact Lie group is a stratified space. Note, however, that in [SL] the stratification is by orbit type, whereas here we use the slightly coarser stratification by infinitesimal orbit type.

Let a circle act on a symplectic manifold M in a Hamiltonian fashion with a moment map $\Phi : M \to \mathbb{R}$. Recall that the **symplectic quotient** (a.k.a. the **reduced space**) is $M_{\text{red}} := \Phi^{-1}(0)/S^1$. If 0 is a regular value for Φ, then the quotient is a symplectic orbifold. More generally, Φ is regular on $M \smallsetminus M^{S^1}$, and $M^r_{\text{red}} := \left(\mu^{-1}(0) \cap (M \smallsetminus M^{S^1})\right)/S^1$ is an orbifold; this is the top stratum. Moreover, since the restriction of the symplectic form on M to $\Phi^{-1}(0) \cap (M \smallsetminus M^{S^1})$ descends to a symplectic form on M^r_{red}, M^r_{red} is naturally oriented.

Recall that the moment map is constant on each component of the fixed point set M^{S^1}, and that these components are isolated. Thus, every component Y of the fixed point set which intersects the zero level set is entirely contained in the level set, and gives rise to a stratum of M_{red} diffeomorphic to Y. To see how these strata fit together, we need the following lemma.

Lemma 3.1. *Let S^1 act on a symplectic manifold (M, ω) in a Hamiltonian fashion with a moment map $\Phi : M \to \mathbb{R}$. Every connected component Y of the fixed point set M^{S^1} has even codimension, say $2n$. Moreover, there exists*

- *positive integers $n_1, \dots, n_k$ such that $\sum_i n_i = n$,*
- *a principal $G := \prod U(n_i) \subset U(n)$ bundle P over Y, and*
- *distinct non-zero weights $\kappa_1, \dots, \kappa_k$,*

such that:

- *there is a diffeomorphism σ from a neighborhood U of Y in M to a neighborhood U_0 of the zero section in the associated bundle $P \times_G \mathbb{C}^n \to Y$;*
- *this diffeomorphism is equivariant with respect to the circle action on $P \times_G \mathbb{C}^n$ defined by the weights κ_i;*
- *the diffeomorphism pulls back the moment map Φ to the map $\mu : P \times_G \mathbb{C}^n \to \mathbb{R}$ given below, i.e., $\Phi \circ \sigma = \mu$, where*

$$\mu([q, (\vec{z}_1, \dots, \vec{z}_k)]) = \frac{1}{2} \sum \kappa_i |\vec{z}_i|^2 + \Phi(Y), \quad \forall\ (\vec{z}_1, \dots, \vec{z}_k) \in \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k} = \mathbb{C}^n.$$

Proof. Consider the symplectic perpendicular bundle $E = TY^\omega$ of Y in (M, ω). Since Y is a symplectic submanifold of M we have $TM|_Y = TY \oplus TY^\omega$. So E is the normal bundle of Y, and E is a symplectic vector bundle. The group S^1 acts on the bundle E by fiber-preserving vector bundle maps. We may choose an S^1 invariant complex structure on E compatible with the symplectic structure. Up to an equivariant homotopy, this complex structure is unique.

A fiber $\mathbb{C}^n$ of E splits into the direct sum of isotypical representations of S^1, $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \mathbb{C}^{n_k}$, so that the action of $\lambda \in S^1$ on $\mathbb{C}^{n_i}$ is given by multiplication by λ^{κ_i} for some weight $\kappa_i \in \mathbb{Z}$. Under the above identification of the fiber of E with $\mathbb{C}^n$ the symplectic structure is the imaginary part of the standard Hermitian inner product. Hence a moment map for the S^1 action on the fiber is

$$\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k} \ni (\vec{z}_1, \dots, \vec{z}_k) \mapsto \frac{1}{2} \sum \kappa_i |\vec{z}_i|^2.$$

The structure group of the vector bundle E reduces to the subgroup of $U(n)$ consisting of transformations that commute with the action of S^1 described above, that is, it reduces to $G := \prod U(n_i) \subset U(n)$. Consequently $E = P \times_G \mathbb{C}^n$ for some principal G bundle P over Y.

The equivariant symplectic embedding theorem (see, for example, Theorem 2.2.1 in [GLS] and the subsequent discussion) implies that we may identify the neighborhood of the submanifold Y in M with a neighborhood of the zero section of E in such a way that a moment map $\mu : E \to \mathbb{R}$ is given by

$$\mu([p, (\vec{z}_1, \dots, \vec{z}_k)]) = \frac{1}{2} \sum \kappa_i |\vec{z}_i|^2 + \text{a constant}$$

for all $([p, (\vec{z}_1, \dots, \vec{z}_k)] \in P \times_G (\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k})$. □

Remark 3.2. Let V^+ and V^- be the the sum of the positive and negative weight spaces, respectively. That is, we may assume that the corresponding weights satisfy $\kappa_1, \dots \kappa_s > 0$ and $\kappa_{s+1}, \dots, \kappa_k < 0$, and set

$$V^+ = \bigoplus_{i=1}^{s} \mathbb{C}^{n_i}, \qquad V^- = \bigoplus_{i=s+1}^{k} \mathbb{C}^{n_i}.$$

We then have linear representations of G on V^+ and V^- so that E splits: $E = E^+ \oplus E^-$ where $E^\pm = P \times_G V^\pm$. By Lemma 3.1, the index of the moment map $\Phi : M \to \mathbb{R}$ at Y is $\dim V^-$.

We now use Lemma 3.1 above to show that the reduced space M_{red} is a simple stratified space.

Proposition 3.3. *Let the circle S^1 act effectively on a compact connected symplectic manifold M in a Hamiltonian fashion with a moment map $\Phi : M \to \mathbb{R}$. Assume that 0 is in the interior of the image of the moment map. The reduced space $M_{red} = \Phi^{-1}(0)/S^1$ is a simple stratified space. The singular strata of M_{red} are connected components Y of the fixed point set M^{S^1} with $\Phi(Y) = 0$. For each such stratum there exists:*

- *a faithful unitary representation $\rho : S^1 \to U(p) \times U(q)$, where p and q are positive integers whose sum is the codimension of Y; and*
- *a principal G bundle $P \to Y$, where G is a subgroup of $U(p) \times U(q)$ which commutes with $\rho(S^1)$;*

so that a neighborhood $\tilde{T}$ of Y in M is the associated cone bundle $P \times_G \overset{\circ}{c}(S^{2p-1} \times_{S^1} S^{2q-1})$.

Remark 3.4. By "a neighborhood $\tilde{T}$ of Y in M_{red} is the associated cone bundle $P \times_G \overset{\circ}{c}(S^{2p-1} \times_{S^1} S^{2q-1})$" we mean that there exists a stratum-preserving homeomorphism from a neighborhood $\tilde{T}$ of Y in M_{red} to the associated bundle $P \times_G \overset{\circ}{c}(S^{2p-1} \times_{S^1} S^{2q-1})$, which restricts to a diffeomorphism on each stratum. The stratification of $P \times_G \overset{\circ}{c}(S^{2p-1} \times_{S^1} S^{2q-1})$ comes from the stratification of the cone $\overset{\circ}{c}(S^{2p-1} \times_{S^1} S^{2q-1})$ into the vertex and the complement of the vertex.

Proof. Let Y be a component of the fixed point set with $\Phi(Y) = 0$. We use the the notation of Lemma 3.1 and Remark 3.2.

By Lemma 3.1 the zero level set $\Phi^{-1}(0)$ near Y is isomorphic to

$$\left\{ [q, (\vec{z}_1, \ldots, \vec{z}_k)] \,\middle|\, \sum_{i=1}^{s} \kappa_i |\vec{z}_i|^2 = \sum_{i=s+1}^{k} \kappa_i |\vec{z}_i|^2 \right\} \simeq P \times_G \overset{\circ}{c}(S^+ \times S^-),$$

where $S^+ = \{z \in V^+ \mid \sum_{i=1}^{s} \kappa_i |\vec{z}_i|^2 = 1\}$, and S^- is defined similarly. Therefore the reduced space $\Phi^{-1}(0)/S^1$ near the stratum Y is $P \times_G \overset{\circ}{c}(S^+ \times_{S^1} S^-)$ where the action of S^1 on $S^+ \times S^- \subset V^+ \times V^-$ is defined by the weights $\kappa_1, \ldots \kappa_k$. ☐

4. The perturbed quotient

In this section we will construct an orbifold $\widetilde{M}_{\text{red}}$, which we call the **perturbed quotient**, together with a map $f : \widetilde{M}_{\text{red}} \to M_{\text{red}}$. The perturbed quotient has two key properties: it is straightforward to explicitly compute its cohomology ring; and f induces a pairing preserving isomorphism between the cohomology ring of the perturbed quotient and the intersection cohomology (middle perversity) of the reduced space. While we do explain what f looks like locally in this section, we defer showing that f induces an isomorphism in cohomology to the last section.

The key idea. The key idea that makes this work is an observation due to Yi Hu (this observation was made in the context of algebraic actions on projective varieties [H]): If 0 is a singular value of an S^1 moment map $\Phi : M \to \mathbb{R}$ and 0 lies in the interior of the image $\Phi(M)$ then for each component Y of the fixed point set M^{S^1} with $\Phi(Y) = 0$ there exists a regular value $\epsilon \in \mathbb{R}$ of Φ and a neighborhood U of Y in M so that there is a natural isomorphism

$$IH^*_{\overline{m}}\left((\Phi^{-1}(0) \cap U)/S^1\right) \simeq H^*\left((\Phi^{-1}(\epsilon) \cap U)/S^1\right).$$

Lemma 4.1. *We use the notation of Lemma 3.1, Remark 3.2 and Proposition 3.3. Fix a component Y of the fixed point set. Consider the associated bundle $P \times_G (V^+ \times V^-)$, together with the moment map*

$$\mu([p, z^+, z^-]) = |z^+|^2 - |z^-|^2,$$

where $z^+ = \sum_{i=1}^{s} \vec{z}_i \in V^+$, $z^- = \sum_{i=s+1}^{k} \vec{z}_i \in V^-$, $|z^+|^2 = \sum_{i=1}^{s} \kappa_i |\vec{z}_i|^2$, and $|z^-|^2 = \sum_{i=s+1}^{k} \kappa_i |\vec{z}_i|^2$.

For any $\epsilon > 0$, $\pm\epsilon$ are regular values of μ and

$$\mu^{-1}(\pm\epsilon)/S^1 = P \times_G X^\pm$$

where

$$X^+ \simeq S^+ \times_{S^1} V^-, \quad \text{a } V^- \text{ bundle over the weighted projective space } S^+/S^1.$$
$$X^- \simeq V^+ \times_{S^1} S^-, \quad \text{a } V^+ \text{ bundle over the weighted projective space } S^-/S^1.$$

Here as in Proposition 3.3, $S^\pm$ is the unit sphere in $V^\pm$, $S^\pm := \{z \in V^\pm \mid |z^\pm|^2 = 1\}$, and the S^1 action on $V^+ \oplus V^-$ is given by the weights $\kappa_1, \ldots, \kappa_k$.

Proof. Note first that the only fixed point in $V^+ \times V^-$ under the S^1 action is $(0, 0)$, and that $\mu(0, 0) = 0$. Therefore any $\alpha \neq 0$ is a regular value of μ. Next assume that Y is a point; in this case $P \times_G (V^+ \times V^-)$ is simply $V^+ \times V^-$. Then $\mu^{-1}(\epsilon) = \{(z^+, z^-) \mid |z^+|^2 - |z^-|^2 = \epsilon\} = \{(z^+, z^-) \mid |z^+|^2 = |z^-|^2 + \epsilon\}$. If $\epsilon > 0$, we have an S^1-equivariant diffeomorphism $S^+ \times V^- \to \mu^{-1}(\epsilon)$ given by $(\zeta, w) \mapsto (\sqrt{\epsilon + |w|^2}\zeta, w)$. Therefore $\mu^{-1}(\epsilon)/S^1 = S^+ \times_{S^1} V^-$ for $\epsilon > 0$. Since the norm $|(z^+, z^-)|^2 = |z^+|^2 + |z^-|^2$ is $G \times S^1$-invariant by construction (see Lemma 3.1 and Remark 3.2) the claim follows. ☐

Observe that $S^+ \times_{S^1} S^-$ is the sphere bundle of the vector bundle $X^+ = S^+ \times_{S^1} V^-$ over the weighted projective space S^+/S^1. Therefore, by collapsing the zero section of the bundle $X^+ \to S^+/S^1$ to a point we obtain a map from X^+ to the cone $\overset{\circ}{c}(S^+ \times_{S^1} S^-)$, which is a diffeomorphism off the zero section onto the cone minus the vertex.

We will see later on that if $\dim V^+ \leq \dim V^-$, then

$$IH^*_{\overline{m}}(\mu^{-1}(0)/S^1) = IH^*_{\overline{m}}(P \times_G \overset{\circ}{c}(S^+ \times_{S^1} S^-)) \simeq H^*(P \times_G X^+) = H^*(\mu^{-1}(\epsilon)/S^1)$$

for any $\epsilon > 0$. Similarly, if $\dim V^+ \geq \dim V^-$ then

$$IH^*_{\overline{m}}(\mu^{-1}(0)/S^1) \simeq H^*(\mu^{-1}(\epsilon)/S^1) \quad \text{for any} \quad \epsilon < 0.$$

In the algebraic category, these isomorphisms follow from the fact that the collapsing map is a **small resolution**, and a fact that small resolutions induce isomorphisms in cohomology. This isomorphism is also valid in the symplectic context, as we prove in the next section.

Note that $\dim V^+ \leq \dim V^-$ if and only if the index of Y as a critical manifold of the Bott-Morse function Φ is at most $\frac{1}{2}(\dim M - \dim Y)$. Unfortunately we cannot expect such inequalities to hold globally, that is, if 0 is a singular value of the moment map $\Phi : M \to \mathbb{R}$ we **should not** expect $IH^*_{\overline{m}}(\Phi^{-1}(0)/S^1) = H^*(\Phi^{-1}(\epsilon)/S^1)$ for some $\epsilon \neq 0$: it may well happen that at one component Y we would need to shift the value of the moment map down and at another component to shift the value up in order to obtain a resolution of the singularities of the reduced space at zero.

4.1. **The Construction.** Let the circle S^1 act effectively on a compact connected symplectic manifold M with a moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of the image $\Phi(M)$. We will now construct a Morse-Bott function $\tilde{\Phi} : M \to \mathbb{R}$ and an S^1 equivariant map $f : \tilde{\Phi}^{-1}(0)/S^1 \to \Phi^{-1}(0)/S^1$ with the following properties.

- The critical points of $\tilde{\Phi}$ are exactly the fixed points of S^1 on M.
- 0 is a regular value of $\tilde{\Phi}$.
- The map $f : \tilde{\Phi}^{-1}(0)/S^1 \to \Phi^{-1}(0)/S^1$ induces an isomorphism in cohomology $IH^*_{\overline{m}}(\Phi^{-1}(0)/S^1) \cong H^*(\tilde{\Phi}^{-1}(0)/S^1)$.

Definition 4.2. We call the subquotient $\widetilde{M}_{\text{red}} := \tilde{\Phi}^{-1}(0)/S^1$ the **perturbed quotient**.

The first two properties guarantee that $\widetilde{M}_{\text{red}}$ is an orbifold, and that it is possible to compute the cohomology ring $H^*(\widetilde{M}_{\text{red}})$ in a fairly straightforward manner. This will be treated explicitly in the next subsection. As we mentioned earlier, the last property will not be proved in this section. However, we will prove Lemma 4.6, which we will later see is sufficient to construct this isomorphism.

For each critical manifold Y_i of Φ in $\Phi^{-1}(0)$, there is a neighborhood U_i of Y_i in M which is equivariantly isomorphic to the model

$$P_i \times_{G_i} (V_i^+ \times V_i^-)$$

where the principal bundle $G_i \to P_i \to Y_i$ and the vector spaces V_i^+, V_i^- are as in the preceding section. We may assume that the U_i's for distinct critical manifolds do not intersect. There exists $\delta > 0$ so that 0 is the only critical value of Φ in $(-\delta, \delta)$ and U_i is the image of the set

$$P_i \times_{G_i} (\{(z_i^+, z_i^-) \mid |z_i^+|^2 + |z_i^-|^2 < 3\delta\}.$$

Therefore, we will simply give our construction on the vector space $V^+ \times V^-$. As long as our definition of $\tilde{\Phi}$ and f are G-invariant, these construction can be naturally extended to the local

model. Additionally, as long as $\tilde{\Phi} = \Phi$ and f is the identity outside the set $\{(z_i^+, z_i^-) \mid |z_i^+|^2 + |z_i^-|^2 < 3\delta\}$, they can be extended globally by taking $\tilde{\Phi} = \Phi$ and $f = \mathrm{id}$ on $M \smallsetminus \cup U_i$.

Choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(t) = 1$ for all $t < \delta$, $\rho(t) = 0$ for all $t > 2\delta$ and $\rho'(t) \le 0$ for all t. Let $C = \sup |\rho'(t)|$, and choose $\epsilon \in \mathbb{R}$ so that $\epsilon \neq 0$, $|\epsilon| < C^{-1}$ and $|\epsilon| < \delta$. Moreover, choose ϵ so that $\epsilon > 0$ if and only if $\dim V^+ \le \dim V^-$. We now define our new function $\tilde{\Phi}$:

$$\tilde{\Phi}(z^+, z^-) := \Phi(z^+, z^-) + \epsilon\rho(|(z^+, z^-)|^2) = |z^+|^2 - |z^-|^2 + \epsilon\rho(|(z^+, z^-)|^2).$$

The norm

$$|(z^+, z^-)|^2 = |z^+|^2 + |z^-|^2$$

is $G \times S^1$-invariant by construction (see Lemma 3.1, Proposition 3.3 and the subsequent discussion). Therefore the function $\rho(|(z^+, z^-)|^2)$, and hence also the function $\tilde{\Phi}$, is $G \times S^1$ invariant. Moreover, for (z^+, z^-) with $|(z^+, z^-)|^2 > 2\delta$, $\tilde{\Phi}(z^+, z^-) = |z^+|^2 - |z^-|^2$. Therefore

$$\tilde{\Phi}^{-1}(0) \cap \{|(z^+, z^-)|^2 > 2\delta\} = \Phi^{-1}(0) \cap \{|(z^+, z^-)|^2 > 2\delta\}.$$

In contrast, for (z^+, z^-) with $|(z^+, z^-)|^2 < \delta$, $\tilde{\Phi}(z^+, z^-) = |z^+|^2 - |z^-|^2 + \epsilon$. Thus $(0, 0)$ is a nondegenerate critical point, and

$$\tilde{\Phi}^{-1}(0) \cap \{|(z^+, z^-)|^2 < \delta\} = \Phi^{-1}(-\epsilon) \cap \{|(z^+, z^-)|^2 < \delta\}.$$

(Note that $|\epsilon| < \delta$ guarantees that $\Phi^{-1}(-\epsilon) \cap \{|(z^+, z^-)|^2 < \delta\} \neq \emptyset$.) Moreover, $(0, 0)$ is the only critical point of $\tilde{\Phi}$, because

$$\begin{aligned} d\tilde{\Phi} &= d|z^+|^2 - d|z^-|^2 + \epsilon d\rho(|(z^+, z^-)|^2) \\ &= \big(1 + \epsilon\rho'(|(z^+, z^-)|^2)\big)\, d|z^+|^2 - \big(1 - \epsilon\rho'(|(z^+, z^-)|^2)\big)\, d|z^-|^2 \end{aligned}$$

and $|1 \pm \epsilon\rho'| \ge 1 - |\epsilon|(\sup |\rho'(t)|) > 0$, since $|\epsilon|(\sup |\rho'(t)|) < 1$ by the choice of ϵ. It follows that $\tilde{\Phi}$ is a Bott-Morse function, and that 0 is a regular value of $\tilde{\Phi}$ (since $\tilde{\Phi}(0, 0) = \epsilon \neq 0$).

Definition 4.3. Let $X = X^r \sqcup \coprod Y_i$ be a simple stratified space. A **resolution** $h : \tilde{X} \to X$ is a continuous surjective map from a smooth orbifold $\tilde{X}$ such that $h^{-1}(X^r)$ is dense in $\tilde{X}$ and $h : f^{-1}(X^r) \to X^r$ is a diffeomorphism.

We will now construct a resolution $f : \widetilde{M}_{\mathrm{red}} \to M_{\mathrm{red}}$. We start by considering a $G \times S^1$-equivariant map $\psi : V^+ \times V^- \to \Phi^{-1}(0)$ defined by

$$\begin{aligned} \psi(z^+, z^-) &= \left(\left(\frac{|z^-|^2}{|z^+|^2} \right)^{1/4} z^+, \left(\frac{|z^+|^2}{|z^-|^2} \right)^{1/4} z^- \right) \quad \text{if} \quad z^+, z^- \neq 0 \\ \psi(0, z^-) &= \psi(z^+, 0) = (0, 0). \end{aligned} \tag{4.4}$$

We let $f : \widetilde{M}_{\mathrm{red}} \to M_{\mathrm{red}}$ be G-equivariant map induced by the restriction $\psi|_{\tilde{\Phi}^{-1}(0)} : \tilde{\Phi}^{-1}(0) \to \Phi^{-1}(0)$.

To prove that f is a resolution, it is enough to show that $\psi|_{\tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0,0)} : \tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0, 0) \to \Phi^{-1}(0) \smallsetminus \{(0, 0)\}$ is a diffeomorphism. It follows from (4.4) that for $(0, 0) \neq (z^+, z^-) \in \Phi^{-1}(0)$,

$$\psi^{-1}(z^+, z^-) = \{(\lambda z^+, \lambda^{-1} z^-) \mid \lambda > 0\}. \tag{4.5}$$

Consequently $\psi|_{\tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0,0)} : \tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0, 0) \to \Phi^{-1}(0) \smallsetminus \{(0, 0)\}$ is one-to-one and onto. Therefore it remains to prove that $d\psi|_{T(\tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0,0))}$ is one-to-one, or, equivalently, that for any $(z^+, z^-) \in \tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0, 0)$

$$0 = \ker d\psi \cap T_{(z^+, z^-)} \tilde{\Phi}^{-1}(0) = \ker d\psi \cap \ker d\tilde{\Phi}.$$

By (4.5), the kernel of $d\psi$ at (z^+, z^-) is spanned by the vector $\frac{d}{d\lambda}\big|_{\lambda=1}(\lambda z^+, \lambda^{-1}z^-)$. Thus it remains to show that for any $(z^+, z^-) \in \tilde{\Phi}^{-1}(0) \smallsetminus \psi^{-1}(0,0)$ we have

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} \tilde{\Phi}(\lambda z^+, \lambda^{-1}z^-) \neq 0.$$

Now

$$\begin{aligned}\frac{d}{d\lambda}\bigg|_{\lambda=1} &\left(|\lambda z^+|^2 - |\lambda^{-1}z^-|^2 + \epsilon\rho(|(\lambda z^+|^2 + |\lambda^{-1}z^-|^2)\right) = \\ &\left(2\lambda|z^+|^2 + 2\lambda^{-3}|z^-|^2 + \epsilon\rho'(|(\lambda z^+|^2 + |\lambda^{-1}z^-|^2)(2\lambda|z^+|^2 - 2\lambda^{-3}|z^-|^2)\right)\big|_{\lambda=1} = \\ &\qquad 2\left(|z^+|^2 + |z^-|^2\right) + 2\epsilon\rho'(|z^+|^2 + |z^-|^2)(|z^+|^2 - |z^-|^2).\end{aligned}$$

For $(z^+, z^-) \in \tilde{\Phi}^{-1}(0)$, we have $|z^+|^2 - |z^-|^2 = -\epsilon\rho(|(z^+, z^-)|)$. Since $\rho'(t) \leq 0$ for all t, $-\epsilon^2\rho(t)\rho'(t) \geq 0$ for all t. Moreover, $(z^+, z^-) \neq (0,0)$. Hence

$$\begin{aligned}\frac{d}{d\lambda}\bigg|_{\lambda=1} &\tilde{\Phi}(\lambda z^+, \lambda^{-1}z^-) = \\ &2\left(|z^+|^2 + |z^-|^2\right) - 2\epsilon\rho'(|z^+|^2 + |z^-|^2)\epsilon\rho(|z^+|^2 + |z^-|^2) \geq 2\left(|z^+|^2 + |z^-|^2\right) > 0.\end{aligned}$$

Thus, we have proved the following.

Lemma 4.6. *Let a circle S^1 act on a symplectic manifold M with a moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of the image $\Phi(M)$. Let $\tilde{\Phi} : M \to \mathbb{R}$ and $f : \widetilde{M}_{red} = \tilde{\Phi}^{-1}(0)/S^1 \to M_{red} = \Phi^{-1}(0)/S^1$ be constructed as above.*

Then f is a resolution. Moreover, for each singular stratum Y of M_{red} there exist:

- *an even dimensional orbifold vector bundle $E \to N$ over a compact orbifold N with a sphere bundle $L \to N$ such that*
$$\dim N \leq \frac{1}{2}\dim E - 1,$$
- *a principal G bundle $P \to Y$,*
- *an action of G on E by vector bundle maps*
- *an isomorphism from a neighborhood of the vertex section of the cone bundle $P \times_G \overset{\circ}{c}(L) \to Y$ to a neighborhood U of Y in M_{red},*
- *an isomorphism from a neighborhood of the zero section of the vector bundle $P \times_G E \to P \times_G N$ to the neighborhood $f^{-1}(U)$ of $f^{-1}(Y)$ in $\widetilde{M}_{red}$ such that the diagram*

$$\begin{array}{ccccc} P \times_G E & \longleftarrow & f^{-1}(U) & \longrightarrow & \widetilde{M}_{red} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ P \times_G \overset{\circ}{c}(L) & \longleftarrow & U & \longrightarrow & M_{red} \end{array}$$

commutes. Here the map h is induced by the natural blow-down map $E \to \overset{\circ}{c}(L)$ taking the zero section to the vertex.

Notice that there is no reason to suspect that the perturbed quotient possesses a symplectic structure.

Remark 4.7. Morally, the above Lemma should be read as a claim that $f : \widetilde{M}_{\rm red} \to M_{\rm red}$ is a small resolution (cf. §6.2 of [GM]).

4.2. **Computation of the cohomology of the perturbed quotient.** We can now compute the cohomology of the perturbed quotient by adapting techniques used to compute the cohomology ring of a symplectic quotient at a regular value.

We begin by reviewing those techniques. Let a circle S^1 act on a compact connected symplectic manifold M with a moment map Φ. Assume that 0 is a regular value. There is a natural restriction from $H^*_{S^1}(M;\mathbb{R})$, the equivariant cohomology of M, to $H^*_{S^1}(\Phi^{-1}(0);\mathbb{R}))$, the equivariant cohomology of the preimage of 0. Since 0 is a regular value, the stabilizer of every point in $\Phi^{-1}(0)$ is discrete. Therefore, there is a natural isomorphism from $H^*_{S^1}(\Phi^{-1}(0),\mathbb{R})$ to the $H^*(M_{\text{red}})$, the ordinary cohomology of the symplectic quotient $M_{\text{red}} : \Phi^{-1}(0)/S^1$. The composition of these two maps gives a natural map, $\kappa : H^*_{S^1}(M) \to H^*(M_{\text{red}})$, called the **Kirwan map**.

Theorem 4.8. *(Kirwan, [K]) Let a circle S^1 act on a compact connected symplectic manifold M with a moment map Φ so that 0 is a regular value. The Kirwan map $\kappa : H^*_{S^1}(M;\mathbb{R}) \to H^*(M_{red};\mathbb{R})$ is surjective.*

Thus, assuming we know the ring structure on M, the ring structure on M_{red} can be computed from the kernel of κ. By Poincare duality, to compute the kernel it is enough to compute the integral of $\kappa(\alpha)$ over the reduced spaces for every equivariant cohomology class α on M. We take one formula for this integral from Kalkman [Ka]; slightly different but morally equivalant formulas were proved by Wu [Wu] and a more general version by Jeffrey-Kirwan [JK]. See also [GK]. All of these results were inspired by a paper of Witten [Wi].

Theorem 4.9. *Let a circle S^1 act on a compact connected symplectic manifold M with a moment map Φ so that 0 is a regular value. Let $\mathcal{F}^+$ denote the set of components F of the fixed point set M^{S^1} such that $\phi(F) > 0$.*

Given an equivariant cohomology class $\alpha \in H^*_{S^1}(M)$, the integral of $\kappa(\alpha)$ over M_{red} is given by the formula

$$\int_{M_{red}} \kappa(\alpha) = \operatorname{Res}_0 \sum_{F\in\mathcal{F}^+} \int_F \frac{i_F^*(\alpha)}{e_F},$$

where e_F denotes the equivariant Euler class of the normal bundle of F.

The right hand side of this formula requires some explanation. The map i_F^* is simply the restriction to F. The equivariant cohomology ring $H^*_{S^1}(F)$, is naturally isomorphic to $H^*(F)[t]$. The equivariant Euler class e_F is invertable in the localized ring $H^*(F)(t)$; thus, $\frac{i_F^*(\alpha)}{e_F}$ is an element of this ring. The integral $\int_F : H^*(F)(t) \to \mathbb{R}(t)$ acts by integrating each coefficient in the series. Finally, Res_0 denotes the operator which returns the coefficient of t^{-1}.

An alternative way of computing the kernel is given by a theorem of Tolman and Weitsman.

Theorem 4.10 (Tolman-Weitsman, [TW])**.** *Let the circle S^1 act on a compact connected symplectic manifold M with a moment map $\Phi : M \to \mathbb{R}$ so that 0 is a regular value.*

Let $\mathcal{F}^+$ denote the set of components F of the fixed point set such that $\Phi(F) > 0$; let $\mathcal{F}^-$ denote the set of components F of the fixed point set such that $\Phi(F) < 0$. Define

$$K_\pm := \{\alpha \in H^*_{S^1}(M) \mid \alpha|_F = 0 \ \ \forall \ \ F \in \mathcal{F}^\pm\}.$$

The kernel of the Kirwan map is $K_+ \oplus K_-$.

In our case, closely analogous propositions are true.

Proposition 4.11. *Let the circle S^1 act on a compact connected symplectic manifold $2n$ dimensional manifold M with a moment map $\Phi : M \to \mathbb{R}$. Assume that 0 is in the interior of $\Phi(M)$. Let $\widetilde{M}_{red}$ denote the perturbed quotient.*

Then there is a surjective ring homomorphism $\kappa : H^*_{S^1}(M) \to H^*(\widetilde{M}_{red})$. Moreover,

- *The kernel of κ is $K_+ \oplus K_-$, where $K_\pm := \{\alpha \in H^*_{S^1}(M) \mid \alpha|_F = 0 \ \forall \ \ F \in \mathcal{F}^\pm\}$.*
- *Given an equivariant cohomology class $\alpha \in H^*_{S^1}(M)$, the integral of $\kappa(\alpha)$ over M_{red} is given by the formula*

$$\int_{M_{red}} \kappa(\alpha) = \mathrm{Res}_0 \sum_{F \in \mathcal{F}^+} \int_F \frac{i_F^*(\alpha)}{e_F},$$

where e_F denotes the equivariant Euler class of the normal bundle of F.

Here, et $\mathcal{F}^+$ denotes the set of components F of the fixed point set M^{S^1} such that either

1. *$\Phi(F) > 0$ or*
2. *$\Phi(F) = 0$ and $2\,\mathrm{index}\, F + \dim F \leq \dim M$.*

Additionally, $\mathcal{F}^-$ denotes all other components of the fixed point set, $\mathcal{F}^- = \mathcal{F} \setminus \mathcal{F}^+$.

The reason that this proposition is true is that the perturbed quotient $\widetilde{M}_{\mathrm{red}}$ is defined in a way very similar to the ordinary reduced space.

Thus, for example, Kalkman's formula follows immediately from the fact that there exists a smooth invariant function $\Phi : M \to \mathbb{R}$ so that 0 is regular and $\widetilde{M}_{\mathrm{red}}$ is defined to be $\tilde{\Phi}^{-1}(0)/S^1$. His proof relies only on the fact that $\tilde{\Phi}^{-1}([0,\infty))$ is a manifold with boundary. Thus, one only need note that $\mathcal{F}^+$ does indeed correspond to the components F of the fixed point set such that $\tilde{\Phi}(F) > 0$.

To see that Kirwan's surjectivity holds, and the Tolman-Weitsman formula for the kernel of κ, we must also use the fact that $\tilde{\Phi}$ is a Morse-Bott function and that its critical points are exactly the fixed points of the action. This is sufficient to prove both theorems, as was pointed out in [TW].

5. The isomorphism

The goal of this section is to prove that the intersection cohomology of the symplectic quotient by a Hamiltonian circle actoin is isomorphic to the (ordinary) cohomology of the perturbed quotient. Because we have already computed the cohomology of the perturbed quotient in Proposition 4.11, this will allow to obtain the description of the intersection cohomology of the symplectic quotient, and thus prove our main theorems. More precisely, we will be done once we have proved the following.

Theorem 5.1. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$ so that 0 is in the interior of $\Phi(M)$. Let $M_{red} := \Phi^{-1}(0)/S^1$ denote the reduced space and let $\widetilde{M}_{red}$ denote the perturbed reduced space. There is a natural pairing preserving isomorphism between the intersection cohomology of the symplectic quotient M_{red} and the cohomology if the perturbed quotient $\widetilde{M}_{red}$.*

More precisely there exists an isomorphism $\psi : H^*(\widetilde{M}_{red}) \to IH^*(M_{red})$ of graded vector spaces such for any $\alpha \in H^p(\widetilde{M}_{red})$ and $\beta \in H^q(\widetilde{M}_{red})$ with $p + q = \dim \widetilde{M}_{red}$ we have

$$\int_{\widetilde{M}_{red}} \alpha \cup \beta = \int_{M_{red}} \langle \psi(\alpha), \psi(\beta) \rangle.$$

Instead of trying to construct the isomorphism between the intersection cohomology of the reduced space and the (ordinary) cohomology of the perturbed quotient directly, we will introduce a new complex $A^*_{\overline{m}}(\widetilde{M}_{\mathrm{red}}) = A^*_{\overline{m}}(f : \widetilde{M}_{\mathrm{red}} \to M_{\mathrm{red}})$ and show that the cohomology of $A^*_{\overline{m}}(\widetilde{M}_{\mathrm{red}} \to M_{\mathrm{red}})$ is naturally isomorphic to both $H^*(\widetilde{M}_{\mathrm{red}})$ and $IH^*_{\overline{m}}(M_{\mathrm{red}})$.

Definition 5.2. Let $f : \tilde{X} \to X$ be a resolution of a simple stratified space. Let X^r be the top stratum of X, $\tilde{X}^r$ be its preimage $f^{-1}(X^r)$, and let $\iota : \tilde{X}^r \hookrightarrow \tilde{X}$ denote the inclusion. By construction, there are maps of complexes $f^* : I\Omega^*_{\overline{m}}(X) \to \Omega^*(\tilde{X}^r)$ and $\iota^* : \Omega^*(\tilde{X}) \to \Omega^*(\tilde{X}^r)$. We define the complex of **resolution forms**

$$A^\bullet_{\overline{m}}(\tilde{X}) = A^\bullet_{\overline{m}}(f : \tilde{X} \to X) := f^*\left(I\Omega^\bullet_{\overline{m}}(X)\right) \cap \iota^*\left(\Omega^\bullet(\tilde{X})\right) \tag{5.3}$$

Note that f^* and ι^* are both injective. Therefore we may think of a resolution form as an intersection form on X^r which extends to a globally defined form on $\tilde{X}$. This gives us the inclusions of complexes $A^\bullet_{\overline{m}}(\tilde{X}) \to I\Omega^\bullet_{\overline{m}}(X)$ and $A^\bullet_{\overline{m}}(\tilde{X}) \to \Omega^\bullet(\tilde{X})$, which induce maps in cohomology $j : H^*(A^\bullet_{\overline{m}}(\tilde{X})) \to IH^*_{\overline{m}}(X)$ and $i : H^*(A^\bullet_{\overline{m}}(\tilde{X})) \to H^*(\tilde{X})$. Note that the graded vector space $H^*(A_{\overline{m}}(\tilde{X})$ has a pairing defined by taking the exterior product of the representatives of the classes and then integrating the product over $\tilde{X}$. Clearly the maps i and j are pairing preserving. Thus, to prove Theorem 5.1 it is enough to show that the maps i and j are isomorphisms.

5.1. **Local issues.**

We start with a simple calculation.

Lemma 5.4. *Let $E \to N$ be an even dimensional orbifold vector bundle over an orbifold N, and let L denote the sphere bundle of E. Then the obvious blow-down map $f : E \to \overset{\circ}{c}(L)$ is a resolution. If $\dim N \leq \frac{1}{2}E - 1$ then the maps*

$$A^\bullet_{\overline{m}}(E) \hookrightarrow \Omega^\bullet(E)$$

and

$$A^\bullet_{\overline{m}}(E) \hookrightarrow I\Omega^\bullet_{\overline{m}}(\overset{\circ}{c}(L))$$

induce isomorphisms in cohomology.

To prove the Lemma we will need the following technical observation.

Lemma 5.5. *Let L be an orbifold. Let α be a closed k form on the cylinder $L \times (0, \infty)$ which vanishes on $L \times (0, a)$ for some a. Then $\alpha = d\beta$ for some $k - 1$ form β which also vanishes on $L \times (0, a)$.*

Proof. Consider first the case where L is a point and $\alpha = f(r)\,dr$ is a 1 form on $(0, \infty)$. Then $\beta = \int_0^r f(s)\,ds$.

In general, if L is not a point, the k-form α has to be of the form $f(r) \wedge dr$ where $f(r)$ is in $\Omega^{k-1}(L)$ for each $r \in (0, \infty)$. Let $\beta = \int_0^r f(s)\,ds$. □

Proof of Lemma 5.4. Note first that the middle perversity of the vertex $*$ of the cone $\overset{\circ}{c}(L)$ is $\overline{m}(*) = \frac{1}{2}\dim E - 1$, since E is even dimensional.

Recall that $\overset{\circ}{c}(L)$ is a stratified space with two strata: the vertex $*$ and the complement $L \times (0, \infty) \simeq E \setminus N$. We may choose the tubular neighborhood T to be any neighborhood of the vertex $*$ of the form $L \times (0, a)/\sim$ for some a.

It follows from the definitions that

$$I\Omega^q_{\overline{m}}(\overset{\circ}{c}(L)) = \begin{cases} \Omega^q(E \setminus N) & \text{for } q < \overline{m}(*) \\ \{\alpha \in \Omega^q(E \setminus N) \mid d\alpha|_T = 0\} & \text{for } q = \overline{m}(*) \\ \{\alpha \in \Omega^q(E \setminus N) \mid \alpha|_T = 0 \text{ and } d\alpha|_T = 0\} & \text{for } q > \overline{m}(*) \end{cases}$$

Consequently

$$A^q_{\overline{m}}(E) = \begin{cases} \Omega^q(E) & \text{for } q < \overline{m}(*) \\ \{\alpha \in \Omega^q(E) \mid d\alpha|_T = 0\} & \text{for } q = \overline{m}(*) \\ \{\alpha \in \Omega^q(E) \mid \alpha|_T = 0 \text{ and } d\alpha|_T = 0\} & \text{for } q > \overline{m}(*) \end{cases}$$

The map $A^q_{\overline{m}}(E) \to I\Omega^q_{\overline{m}}(\overset{\circ}{c}(L))$ is induced by the restriction from $\Omega^*(E)$ to $\Omega^*(E \smallsetminus N)$.

It follows from Lemma 5.5 that for $q \leq \overline{m}(*)$ the map $H^q(A^\bullet_{\overline{m}}(E)) \to H^q(E)$ is an isomorphism and that $H^q(A^\bullet_{\overline{m}}(E)) = 0$ for $q > \overline{m}(*)$. Since $H^*(E) = H^*(N)$ and since $\overline{m}(*) \geq \dim N$ by assumption, the map $H^q(A^\bullet_{\overline{m}}(E)) \to H^q(E)$ is an isomorphism for all q.

Similarly,

$$IH^q_{\overline{m}}(\overset{\circ}{c}(L)) = \begin{cases} H^q(E \smallsetminus N) & \text{for } q \leq \overline{m}(*) \\ 0 & \text{for } q > \overline{m}(*) \end{cases}.$$

Consider the Gysin sequence

$$\cdots \to H^{q-\lambda}(E) \to H^q(E) \to H^q(E \smallsetminus N) \to H^{q-\lambda+1}(E) \to \cdots$$

where $\lambda = \dim E - \dim N$. Since for $q - \lambda + 1 \leq -1$ (i.e., for $q \leq \lambda - 2$) we have $H^{q-\lambda+1}(E) = 0 = H^{q-\lambda}(E)$, the pull-back map $H^q(E) \to H^q(E \smallsetminus N)$ is an isomorphism. In particular the pull-back is an isomorphism for for $q \leq \overline{m}(*) = \frac{1}{2}\dim E - 1 = \dim E - (\frac{1}{2}\dim E - 1) - 2 \leq \dim E - \dim N - 2 = \lambda - 2$. $\square$

Proposition 5.6. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$. Assume that 0 is in the interior of the image $\Phi(M)$. Let $M_{red} := \Phi^{-1}(0)/S^1$ denote the reduced space and let $f : \widetilde{M}_{red} \to M_{red}$ denote its resolution by the perturbed quotient.*

There exists a cover $\mathcal{U}$ of M_{red} such that the natural inclusions $A^*_{\overline{m}}(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k})) \to I\Omega^*_{\overline{m}}(U_{\alpha_1} \cap \cdots U_{\alpha_k})$ and $A^*_{\overline{m}}(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k})) \to \Omega^*(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k}))$ induce isomorphisms in cohomology for all k-tuples $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ of elements of $\mathcal{U}$.

Proof. We have seen in Proposition 3.3 that $M_{\text{red}} = M^r_{\text{red}} \coprod Y_i$ where Y_i are compact manifolds. Further, for each singular stratum Y there exists a tubular neighborhood $\tilde{T}$ of Y, the fiber bundle $\overset{\circ}{c}(L) \to \tilde{T} \xrightarrow{\pi} Y$, and the map $r : \tilde{T} \to [0, 1)$ (c.f. Remark 2.2). Recall also that by Lemma 4.6 we may assume that $\tilde{T} = P \times_G \overset{\circ}{c}(L)$, that $f^{-1}(\tilde{T}) = P \times_G E$ and that $f : P \times_G E \to P \times_G \overset{\circ}{c}(L)$ is induced by the obvious blow-down map $E \to \overset{\circ}{c}(L)$.

We take $U_0 = M_{\text{red}} \smallsetminus \cup r_i^{-1}([0, 1/2]) = M_{\text{red}} \smallsetminus \overline{T}_i$. Since each singular stratum Y is a compact manifold, it possesses a finite good cover $\{V_\alpha\}$. Moreover we may assume that $\pi^{-1}(V_\alpha) \simeq \overset{\circ}{c}(L) \times V_\alpha$. We take $U_\alpha := \pi^{-1}(V_\alpha) \cap \tilde{T} \subset \tilde{T}$. This give us a cover $\mathcal{U}$ of M_{red}.

Note that by construction for a k-tuple $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ of elements of $\mathcal{U}$ we either have that $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ does not intersect any singular stratum Y (in which case $f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k})$ and $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ are diffeomorphic) or there is a unique stratum Y such that $Y \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \neq \emptyset$. In the latter case $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \simeq D \times \overset{\circ}{c}(L)$, $f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k}) \simeq D \times E$ and $f : f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k}) \to U_{\alpha_1} \cap \cdots U_{\alpha_k}$ is equivalent to the map $h \times id : E \times D \to \overset{\circ}{c}(L) \times D$, where D is a disk in Y and $h : E \to \overset{\circ}{c}(L)$ is the resolution.

Given a disk D and a set X we have an inclusion $\iota : X \hookrightarrow X \times D$, $\iota(x) = (x, 0)$. Clearly the diagram

$$\begin{array}{ccc} E & \xrightarrow{\iota} & E \times D \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h\times id} \\ \mathring{c}(L) & \xrightarrow{\iota} & \mathring{c}(L) \times D \end{array}$$

commutes. Since $h : E \to \mathring{c}(L)$ and $h \times id : E \times D \to \mathring{c}(L) \times D$ are resolutions, we have a commutative diagram of complexes

$$\begin{array}{ccccc} \Omega^*(E) & \xleftarrow{\iota^*} & \Omega^*(E \times D) & = & \Omega^*(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k})) \\ \uparrow & & \uparrow & & \\ A^*_{\overline{m}}(E) & \xleftarrow{\iota^*} & A^*_{\overline{m}}(E \times D) & = & A^*_{\overline{m}}(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_k})) \\ \downarrow & & \downarrow & & \\ I\Omega^*_{\overline{m}}(\mathring{c}(L)) & \xleftarrow{\iota^*} & I\Omega^*_{\overline{m}}(\mathring{c}(L) \times D) & = & I\Omega^*_{\overline{m}}(U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}). \end{array} \tag{5.7}$$

Since the disk D is contractible, the horizontal maps induce isomorphisms in cohomology. Since the left vertical maps induced isomorphisms in cohomology by Lemma 5.4, the right vertical maps induce isomorphisms as well. $\square$

Proposition 5.8. *Let the circle S^1 act on a compact connected symplectic manifold M with moment map $\Phi : M \to \mathbb{R}$. Assume that 0 is in the interior of the image $\Phi(M)$. Let $M_{red} := \Phi^{-1}(0)/S^1$ denote the reduced space and let $f : \widetilde{M}_{red} \to M_{red}$ denote its resolution by the perturbed quotient.*

The inclusions $A^*_{\overline{m}}(\widetilde{M}_{red}) \to I\Omega^*_{\overline{m}}(M_{red})$ and $A^*_{\overline{m}}(\widetilde{M}_{red}) \to \Omega^*(\widetilde{M}_{red})$ induce isomorphisms in cohomology.

Proof. The proof is now a standard spectral sequence argument.

Let $\mathcal{U} = \{U_\alpha\}$ be the cover of M_{red} constructed in the proof of Proposition 5.6. We now construct a continuous partition of unity ρ_α subordinate to the cover $\mathcal{U}$ with the properties that

- the functions ρ_α restrict to smooth functions on M^r_{red},
- the functions ρ_α are constant along the fibers of $\pi : T \to Y$ for all the singular strata Y.

These properties ensure that

- $\{f^*\rho_\alpha\}$ is a partition of unity on $\widetilde{M}_{\text{red}}$ subordinate to the cover $\{f^{-1}(U_\alpha)\}$ of $\widetilde{M}_{\text{red}}$, and that
- for any intersection form $\gamma \in I\Omega^*_{\overline{m}}(M_{\text{red}})$ or resolution form $\delta \in A^*_{\overline{m}}$, the products $\rho_\alpha\gamma$ and $f^*\rho_\alpha\delta$ are also in $I\Omega^*_{\overline{m}}(M_{\text{red}})$ and $A^*_{\overline{m}}$, respectively.

We first consider the set U_0 which is entirely contained in the smooth part M^r_{red} of the quotient. We choose $\tilde{\rho}_0$ to be a smooth nonnegative function on M^r_{red} supported in U_0 with $\tilde{\rho}_0 = 1$ in $M_{\text{red}} \smallsetminus \cup r_i^{-1}([0, 3/4))$.

By construction a set U_α with $U_\alpha \cap Y \neq \emptyset$ is of the form $\pi^{-1}(V_\alpha) \cap \tilde{T}$ where $\{V_\alpha\}$ is a good cover of Y. We can choose a smooth partition of unity $\{\tau_\alpha\}$ on Y subordinate to $\{V_\alpha\}$ and also a nonnegative smooth function σ on M^r_{red} supported in $\tilde{T}$ with σ identically 1 on the $\overline{T} \cap M^r_{\text{red}} = r^{-1}([0, 1/2]) \cap M^r_{\text{red}}$. Let $\tilde{\rho}_\alpha := \sigma(\pi^*\tau_\alpha|_{M_{\text{red}}})$; it extends to a continuous function on M_{red}. The functions $\rho_\alpha := \frac{\tilde{\rho}_\alpha}{\sum_\beta \tilde{\rho}_\beta}$ form the desired partition of unity.

Next we define three double complexes whose i, j'th terms are given as follows for $j \geq 0$:

$$A^{i,j}_{\overline{m}}(\mathcal{U}) := \oplus A^{i}_{\overline{m}}(f^{-1}(U_{\alpha_0}) \cap \cdots \cap f^{-1}(U_{\alpha_j}))$$

$$I\Omega^{i,j}_{\overline{m}}(\mathcal{U}) := \oplus I\Omega^{i}_{\overline{m}}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_j}), \quad \text{and}$$

$$\tilde{\Omega}^{i,j}(\mathcal{U}) := \oplus \Omega^{i}(f^{-1}(U_{\alpha_0}) \cap \cdots \cap f^{-1}(U_{\alpha_j})),$$

where the sums are taken over all $j+1$-tuples $\{\alpha_0, \ldots \alpha_j\}$. For all three complexes the differentials are given by the de Rham and Čech differentials.

First, in order to show that the cohomology of the double complexes are the intersection cohomology of M_{red}, the cohomology of the complex $A^{i}_{\overline{m}}(\widetilde{M}_{\text{red}})$ and the cohomology of $\widetilde{M}_{\text{red}}$ respectively, we will consider the spectral sequences associated to the filtration by i. Since we constructed a nice partition of unity subordinate to the locally finite cover $\mathcal{U}$, for any fixed i the Čech cohomology of the sheaf $I\Omega^{i}_{\overline{m}}$ is trivial for $j > 0$, and for $j = 0$ consists of the global forms $I\Omega^{i}_{\overline{m}}(M_{\text{red}})$. Thus, the spectral sequence converges at the E_2 term. Moreover, $E_2^{i,j} = 0$ for $j > 0$, and $E_2^{i,0} = IH^{i}_{\overline{m}}(\widetilde{M}_{\text{red}})$ for all i. Thus, the cohomology of the double complex $I\Omega^{i,j}_{\overline{m}}(\mathcal{U})$ is the intersection cohomology $IH^{i}_{\overline{m}}(M_{\text{red}})$. Virtually identical arguments show that the cohomology of the double complexes $A^{i,j}_{\overline{m}}(\mathcal{U})$ and $\tilde{\Omega}^{i,j}(\mathcal{U})$ are the cohomology of the complexes $A^{i}_{\overline{m}}(\widetilde{M}_{\text{red}})$ and $\Omega^{i}(\widetilde{M}_{\text{red}})$, respectively.

Next, in order to show that inclusions induce isomorphism from $H^*(A_{\overline{m}}(\widetilde{M}_{\text{red}}))$ to $IH^{*}_{\overline{m}}(M_{\text{red}})$ and from $H^*(A_{\overline{m}}(\widetilde{M}_{\text{red}}))$ to $H^*(\widetilde{M}_{\text{red}})$ respectively, we will consider the spectral sequences associated with the filtration by j. The double complex $A^{i,j}_{\overline{m}}(\mathcal{U}) := \oplus A^{i}_{\overline{m}}(f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_j}))$ includes naturally into the double complex $I\Omega^{i,j}_{\overline{m}}(\mathcal{U}) := \oplus I\Omega^{i}_{\overline{m}}(U_{\alpha_1} \cap \cdots \cap U_{\alpha j})$. By Proposition 5.6 this inclusion induces an isomorphism on the E_1 terms of these spectral sequences. This implies that the inclusion induces an isomorphism on every E_k. Hence, by the proceeding paragraph, inclusion induces an isomorphism from $H^*(A_{\overline{m}}(\widetilde{M}_{\text{red}}))$ to $IH^{*}_{\overline{m}}(M_{\text{red}})$. An essentially identical argument shows that the inclusion $A^{*}_{\overline{m}}(\widetilde{M}_{\text{red}}) \to \Omega^*(\widetilde{M}_{\text{red}})$ induces an isomorphism in cohomology. □

This completes the proof of Theorem 5.1. By combining Theorem 5.1 with Proposition 4.11 we now obtain the main result of the paper: Theorem 1 and Theorem 1′.

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